

Proof of Factorization of J/Ψ Production in Non-Equilibrium QCD at RHIC and LHC

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Abstract

J/ψ suppression/production is one of the main signature of quark-gluon plasma detection at RHIC and LHC. In order to study j/ψ suppression/production in high energy heavy-ion collisions at RHIC and LHC, one needs to prove the factorization theorem of j/ψ production in non-equilibrium QCD medium, otherwise one will predict infinite cross section of j/ψ . In this paper we prove factorization theorem of j/ψ production in non-equilibrium QCD at RHIC and LHC at all order in coupling constant.

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I. INTRODUCTION

Just after $\sim 10^{-12}$ seconds of the big bang, the universe was filled with a state of matter known as quark-gluon plasma. The quark-gluon plasma is much hotter and denser than the ordinary matter we see today. The temperature of the quark-gluon plasma is $\gtrsim 200$ MeV ($\gtrsim 3.2 \times 10^{12}$ K) which is about million times larger than the temperature of the sun ($\sim 1.56 \times 10^7$ K). The quark-gluon plasma is much denser than neutron stars, besides black holes, there's nothing denser than this. Hence it is important to recreate this early universe scenario in the laboratory, *i. e.*, to produce quark-gluon plasma in the laboratory. At present RHIC (relativistic heavy-ion colliders) at BNL and LHC (large hadron collider) at CERN are the best facilities to produce quark-gluon plasma in the laboratory [1, 2].

RHIC and LHC use high energy heavy-ion collisions to produce quark-gluon plasma. RHIC collides two gold nuclei at $\sqrt{s} = 200$ GeV per nucleon [3, 4] with total energy ~ 40 TeV. The LHC (in its first run) collides two lead nuclei at $\sqrt{s} = 2.76$ TeV per nucleon [5–7] with total energy ~ 574 TeV. Since these huge total energies are deposited in very small volume at the initial moment of the nuclear collisions at RHIC and LHC, the energy density necessary to create temperature ~ 200 MeV to produce quark-gluon plasma might have been achieved at RHIC and LHC. The LHC (in its second run) will collide two lead nuclei at $\sqrt{s} = 5.5$ TeV per nucleon with total energy ~ 1150 TeV which will be even higher.

However, since the two nuclei at RHIC and LHC travel almost at speed of light, the quark-gluon plasma produced at RHIC and LHC may be in non-equilibrium. This is because the partons inside the nuclei at RHIC and LHC carry very high longitudinal momentum and very small transverse momentum just before the nuclear collision. Since the hadronization takes place in a very short time scale (the typical time scale for hadronization in QCD is $\sim 10^{-24}$ seconds), there may not be enough time for these highly non-isotropic partons to thermalize at RHIC and LHC. Hence, in order to make meaningful comparison of the theory with the experimental data on hadron production it may be necessary to study nonequilibrium-nonperturbative QCD at RHIC and LHC. This, however, is a difficult problem.

Since quarks and gluons are not directly observed we can not directly detect the quark-gluon plasma at RHIC and LHC. Hence indirect signatures are proposed for the detection of quark-gluon plasma at RHIC and LHC. The main signatures of quark-gluon plasma detection are 1) j/ψ suppression/production 2) dilepton production 3) direct photon production, 4)

strangeness enhancement and 5) jet quenching etc.. In this paper we will focus on the j/ψ suppression/production signature for the quark-gluon plasma detection at RHIC and LHC.

Note that the j/ψ suppression was predicted to be a signature of quark-gluon plasma detection by using lattice QCD calculation at finite temperature in equilibrium [8]. It was argued that at high temperature the Debye screening length is much smaller than the j/ψ radius, leading to complete suppression of j/ψ production in quark-gluon plasma. However, this prediction uses lattice QCD calculation at finite temperature which may not be applicable at RHIC and LHC where the quark-gluon plasma may be in non-equilibrium. Hence, in order to make comparison of the theory with the experimental data on j/ψ production at RHIC and LHC, it may be necessary to study j/ψ production in non-equilibrium QCD.

The j/ψ suppression is experimentally observed in Au-Au collisions at $\sqrt{s} = 200$ GeV at RHIC [9, 10] and in Pb-Pb collisions at $\sqrt{s} = 2.76$ TeV at LHC (in its first run) [11–13]. It is interesting to note that in most central heavy-ion collisions the PHENIX collaboration [9] at RHIC has measured more j/ψ suppression than the ALICE collaboration [13] at LHC (in its first run). In order to explain this it is argued that there is j/ψ enhancement [14] and charm recombination [15] in QCD medium etc., however these approaches need to incorporate non-perturbative QCD mechanism for heavy quarkonium production from $c\bar{c}$ pair. Hence the determination of in-medium properties of heavy quarkonium at RHIC and LHC remains a challenging theoretical task.

The experimental data at RHIC and LHC suggest that the j/ψ production cross section is modified in the heavy-ion collisions in comparison to the p-p collisions at the same center of mass energy. This modification can be due to the cold nuclear matter effects such as nuclear shadowing, saturation, energy loss and quark-antiquark breakup etc. and due to the hot QCD medium effects such as color screening and secondary charm production etc.. The cold nuclear matter effects are studied both theoretically and experimentally, for example by using p-A collisions at the corresponding center of mass energy. The hot QCD medium effects such as color screening is well studied theoretically by using lattice QCD at finite temperature [8, 17], but needs to be extended to non-equilibrium QCD to be applicable at RHIC and LHC. Similarly secondary charm production is well studied at finite temperature [14, 15, 18, 19], but how these secondary charm-anticharm produces j/ψ by using non-perturbative QCD mechanism is not well understood, which also needs to be extended to non-equilibrium QCD at RHIC and LHC.

Hence the j/ψ production mechanism in QCD in vacuum (for example in p-p collisions) need to be modified to include QCD medium effects at the heavy-ion collisions at RHIC and LHC. As mentioned above, since the two nuclei at RHIC and LHC travel almost at speed of light, the quark-gluon plasma at RHIC and LHC may be in non-equilibrium. Hence, in order to detect quark-gluon plasma by using j/ψ as a signature, one needs to study j/ψ production in non-equilibrium QCD at RHIC and LHC.

Note that in order to study j/ψ production in p-p collisions at high energy one needs to prove factorization theorem. The need to prove factorization theorem arises because in the absence of the proof of factorization theorem one will predict infinite cross section of j/ψ production [20–28]. This is due to the interaction between charm (anticharm) quark with the nearby light quark or gluon. The soft gluon exchange between charm (anticharm) quark and the nearby light quark (or gluon) gives infrared divergence which makes the partonic level cross section infinite. Hence it is important to prove that the effect of such infrared (soft) gluon exchanges either cancel or the non-canceling infrared divergences be absorbed into the definition of the non-perturbative matrix element of the j/ψ because (soft) infrared regime corresponds to non-perturbative QCD.

In the non-equilibrium QCD medium at RHIC and LHC there are many more nearby light quarks and gluons (in comparison to the corresponding situation in p-p collisions in QCD in vacuum) with which the charm (anticharm) quark interacts. Because of this, infrared divergences seem to be severe at RHIC and LHC heavy-ion collisions than that in corresponding p-p collisions. Hence it is necessary to prove that the infrared divergences due to the soft gluons exchange between charm (anticharm) quark and the nearby light quark (or gluon) in non-equilibrium QCD medium at RHIC and LHC either cancel or the non-canceling infrared divergences be absorbed into the definition of the non-perturbative matrix element of the j/ψ production in non-equilibrium QCD.

Since the j/ψ suppression as a signature of quark-gluon plasma detection was suggested by using j/ψ from color singlet $c\bar{c}$ pair in [8], by comparing with the correlation length obtained from lattice QCD [29], we will consider j/ψ production from color singlet $c\bar{c}$ pair in non-equilibrium QCD in this paper. We will prove, in this paper, that the infrared divergences due to the soft gluons exchange between charm quark and the nearby light-like quark (or gluon) in non-equilibrium QCD at RHIC and LHC exactly cancel with the corresponding infrared divergences due to the soft gluons exchange between anticharm quark and the same

nearby light-like quark (or gluon) at all order in coupling constant in the j/ψ production from color singlet $c\bar{c}$ pair in non-equilibrium QCD. This proves the factorization theorem of j/ψ production in non-equilibrium QCD at RHIC and LHC at all order in coupling constant.

The paper is organized as follows. In section II we briefly discuss path integral formulation of non-equilibrium QCD. In section III we discuss the non-perturbative matrix element of j/ψ production from color singlet charm-anticharm pair in non-equilibrium QCD. In section IV we discuss how longitudinal polarization of the gauge field in quantum field theory is used to describe infrared divergence. In section V we discuss the gauge field generated by the eikonal current of the light-like Wilson line in quantum field theory. In section VI we prove factorization theorem of j/ψ production at high energy colliders. In section VII we prove factorization theorem of j/ψ production in non-equilibrium QCD at RHIC and LHC. Section VIII contains conclusions.

II. NON-EQUILIBRIUM QCD USING CLOSED-TIME PATH INTEGRAL FORMALISM

Unlike pp collisions, the ground state at RHIC and LHC heavy-ion collisions (due to the presence of a QCD medium at initial time $t = t_{in}$ (say $t_{in}=0$) is not a vacuum state $|0\rangle$ any more. We denote $|in\rangle$ as the initial state of the non-equilibrium QCD medium at t_{in} .

Consider massless scalar field theory first. In the closed-time path (CTP) formalism the generating functional in the path integral formulation for scalar field theory in non-equilibrium is given by

$$Z[\rho, J_+, J_-] = \int [d\phi_+] [d\phi_-] \exp[i[S[\phi_+] - S[\phi_-] + \int d^4x J_+ \phi_+ - \int d^4x J_- \phi_-]] \langle \phi_+, 0 | \rho | 0, \phi_- \rangle \quad (1)$$

where ρ is the initial density of state, $S[\phi]$ is the full action in scalar field theory and $|\phi_\pm, 0\rangle$ is the quantum state corresponding to the field configuration $\phi_\pm(\vec{x}, t=0)$. Since there are two sources J_+ and J_- corresponding to two time branches $+$ and $-$ in non-equilibrium quantum field theory one finds that there are four Green's functions in non-equilibrium. The four Green's functions in non-equilibrium can be obtained from the generating functional as follows

$$G_{++}(x, x') = \frac{\delta Z[\rho, J_+, J_-]}{i^2 \delta J_+(x) \delta J_+(x')} = \langle in | T \phi(x) \phi(x') | in \rangle$$

$$\begin{aligned}
G_{--}(x, x') &= \frac{\delta Z[\rho, J_+, J_-]}{(-i)^2 \delta J_-(x) J_-(x')} = \langle in | \bar{T} \phi(x) \phi(x') | in \rangle \\
G_{+-}(x, x') &= \frac{\delta Z[\rho, J_+, J_-]}{-i^2 \delta J_+(x) J_-(x')} = \langle in | \phi(x') \phi(x) | in \rangle \\
G_{-+}(x, x') &= \frac{\delta Z[\rho, J_+, J_-]}{-i^2 \delta J_-(x) J_+(x')} = \langle in | \phi(x) \phi(x') | in \rangle
\end{aligned} \tag{2}$$

where $+$ ($-$) sign corresponds to upper(lower) time branch of the Schwinger-Keldysh closed-time path [51, 52], T is the time order product and \bar{T} is the anti-time order product.

The generating functional in non-equilibrium QCD (including heavy quark) in the path integral formulation is given by [53, 54]

$$\begin{aligned}
& Z[\rho, J_+, J_-, \eta_+^u, \eta_-^u, \bar{\eta}_+^u, \bar{\eta}_-^u, \eta_+^d, \eta_-^d, \bar{\eta}_+^d, \bar{\eta}_-^d, \eta_+^s, \eta_-^s, \bar{\eta}_+^s, \bar{\eta}_-^s, \eta_+^h, \eta_-^h, \bar{\eta}_+^h, \bar{\eta}_-^h] \\
&= \int [dQ_+][dQ_-][d\bar{\psi}_{1+}][d\bar{\psi}_{1-}][d\psi_{1+}][d\psi_{1-}][d\bar{\psi}_{2+}][d\bar{\psi}_{2-}][d\psi_{2+}][d\psi_{2-}][d\bar{\psi}_{3+}][d\bar{\psi}_{3-}][d\psi_{3+}][d\psi_{3-}] \\
& [d\bar{\Psi}_+][d\bar{\Psi}_-][d\Psi_+][d\Psi_-] \times \det\left(\frac{\delta \partial_\mu Q_+^{\mu a}}{\delta \omega_+^b}\right) \times \det\left(\frac{\delta \partial_\mu Q_-^{\mu a}}{\delta \omega_-^b}\right) \\
& \exp\left[i \int d^4x \left[-\frac{1}{4}(F_{\mu\nu}^a[Q_+] - F_{\mu\nu}^a[Q_-]) - \frac{1}{2\alpha}((\partial_\mu Q_+^{\mu a})^2 - (\partial_\mu Q_-^{\mu a})^2)\right.\right. \\
& + \sum_{l=1}^3 \bar{\psi}_{l+}[i\gamma^\mu \partial_\mu - m_l + gT^a \gamma^\mu Q_{\mu+}^a]\psi_{l+} - \sum_{l=1}^3 \bar{\psi}_{l-}[i\gamma^\mu \partial_\mu - m_l + gT^a \gamma^\mu Q_{\mu-}^a]\psi_{l-} \\
& + \bar{\Psi}_+[i\gamma^\mu \partial_\mu - M + gT^a \gamma^\mu Q_{\mu+}^a]\Psi_+ - \bar{\Psi}_-[i\gamma^\mu \partial_\mu - M + gT^a \gamma^\mu Q_{\mu-}^a]\Psi_- + J_+ \cdot Q_+ - J_- \cdot Q_- \\
& + \sum_{l=1}^3 [\bar{\eta}_{l+} \cdot \psi_{l+} - \bar{\eta}_{l-} \cdot \psi_{l-} + \bar{\psi}_{l+} \cdot \eta_{l+} - \bar{\psi}_{l-} \cdot \eta_{l-}] + \bar{\eta}_{h+} \cdot \Psi_+ - \bar{\eta}_{h-} \cdot \Psi_- + \bar{\Psi}_+ \cdot \eta_{h+} - \bar{\Psi}_- \cdot \eta_{h-}] \\
& \left. \times \langle Q_+, \psi_+^u, \bar{\psi}_+^u, \psi_+^d, \bar{\psi}_+^d, \psi_+^s, \bar{\psi}_+^s, \Psi_+, \bar{\Psi}_+, 0 | \rho | 0, \Psi_-, \bar{\Psi}_-, \bar{\psi}_-^s, \psi_-^s, \bar{\psi}_-^d, \psi_-^d, \bar{\psi}_-^u, \psi_-^u, Q_- \rangle\right]
\end{aligned} \tag{3}$$

where ρ is the initial density of state. The state $|Q^\pm, \psi^\pm, \bar{\psi}^\pm, 0\rangle$ corresponds to the field configurations $Q_\mu^a(\vec{x}, t = t_{in} = 0)$, $\psi(\vec{x}, t = t_{in} = 0)$ and $\bar{\psi}(\vec{x}, t = t_{in} = 0)$ respectively and $J^{\mu a}(x)$ is the external source for the quantum gluon field $Q^{\mu a}(x)$ and the $\bar{\eta}^u$, $\bar{\eta}^d$, $\bar{\eta}^s$ are external sources for $l = 1, 2, 3 = u, d, s$ quark fields respectively and $\bar{\eta}_h$ is the external source for the heavy quark field Ψ , m_l is the mass of the light quark of type l , the M is the mass of the heavy quark and

$$F_{\mu\nu}^a[Q] = \partial_\mu Q_\nu^a(x) - \partial_\nu Q_\mu^a(x) + gf^{abc}Q_\mu^b(x)Q_\nu^c(x), \quad F_{\mu\nu}^{a2}[Q] = F^{\mu\nu a}[Q]F_{\mu\nu}^a[Q]. \tag{4}$$

Note that we work in the frozen ghost formalism [53, 54] for the medium part at the initial time $t = t_{in} = 0$.

For the heavy quark Dirac field $\Psi_r(x)$ in non-equilibrium QCD, the nonequilibrium-nonperturbative matrix element of the type $\langle in | \bar{\Psi}_r(x_1) \Psi_r(x_1) \bar{\Psi}_s(x_2) \Psi_s(x_2) | in \rangle$ is given by [25]

$$\begin{aligned} & \langle in | \bar{\Psi}_r(x_1) \Psi_r(x_1) \bar{\Psi}_s(x_2) \Psi_s(x_2) | in \rangle \\ &= \frac{\delta}{\delta \eta_{hr}(x_1)} \frac{\delta}{\delta \bar{\eta}_{hr}(x_1)} \frac{\delta}{\delta \eta_{hs}(x_2)} \frac{\delta}{\delta \bar{\eta}_{hs}(x_2)} Z[\rho, J_+, J_-, \eta_+^u, \eta_-^u, \bar{\eta}_+^u, \bar{\eta}_-^u, \eta_+^d, \eta_-^d, \bar{\eta}_+^d, \bar{\eta}_-^d, \eta_+^s, \eta_-^s, \bar{\eta}_+^s, \bar{\eta}_-^s, \\ & \quad , \eta_+^h, \eta_-^h, \bar{\eta}_+^h, \bar{\eta}_-^h] |_{J_+=J_-=\eta_+^u=\eta_-^u=\bar{\eta}_+^u=\bar{\eta}_-^u=\eta_+^d=\eta_-^d=\bar{\eta}_+^d=\bar{\eta}_-^d=\eta_+^s=\eta_-^s=\bar{\eta}_+^s=\bar{\eta}_-^s=\eta_+^h=\eta_-^h=\bar{\eta}_+^h=\bar{\eta}_-^h=0} \end{aligned} \quad (5)$$

where the repeated (closed-time path) indices $r, s = +, -$ are not summed and the suppression of the normalization factor $Z[0]$ is understood as it will cancel in the final result, see eq. (82).

III. NON-PERTURBATIVE MATRIX ELEMENT OF J/Ψ PRODUCTION FROM COLOR SINGLET CHARM-ANTICHARM PAIR IN NON-EQUILIBRIUM QCD

The charm-anticharm quark ($c\bar{c}$) pair production in the initial collision of the two nuclei at RHIC and LHC can be calculated by using pQCD. The spectral notation of j/ψ is given by $^{2S+1}L_J$ where spin quantum number $S = 1$ (spin triplet), orbital angular momentum quantum number $L = 0$ (S -wave) and total angular momentum quantum number $J = 1$. Once the $c\bar{c}$ pair is formed in the color singlet and spin triplet state it can form j/ψ . The formation of j/ψ from $c\bar{c}$ involves non-perturbative QCD mechanism which can not be calculated by using pQCD.

Note that the potential model calculations [30] use phenomenological potentials like Coulomb potential or Coulomb plus linear potential etc. between color singlet $c\bar{c}$ by solving non-relativistic wave equation to determine the radial wave function at the origin $R(0)$ for the j/ψ formation from color singlet $c\bar{c}$ pair. However, strictly speaking, unlike QED, the exact form of the potential energy between $c\bar{c}$ pair is not known in QCD because we do not know the exact form of the classical Yang-Mills potential $A^{\mu a}(x)$ yet, although Yang-Mills theory was discovered almost 60 years ago. Hence, unlike QED, we depend on the experimental data in QCD [30, 31] to determine the non-perturbative matrix element of the heavy quarkonium production from heavy quark and antiquark pair. The non-perturbative matrix element for j/ψ formation from color singlet $c\bar{c}$ pair needs to be universal, *i. e.*, it should not change from one experiment to other. Hence it is necessary to use the correct definition

of the non-perturbative matrix element of j/ψ production from color singlet and spin triplet $c\bar{c}$ pair. If one does not use the correct definition of the non-perturbative matrix element of j/ψ production from color singlet and spin triplet $c\bar{c}$ pair then one will predict infinite cross section for j/ψ production. This can be seen as follows.

If the factorization theorem holds, then the production cross section for j/ψ at transverse momentum P_T can be written as the product of perturbative cross section of $c\bar{c}$ production times the universal non-perturbative matrix element,

$$d\sigma_{A+B \rightarrow j/\psi + X(P_T)} = d\hat{\sigma}_{A+B \rightarrow c\bar{c}[^3S_1] + X(P_T)} < 0 | \mathcal{O}_{j/\psi} | 0 > \quad (6)$$

where $d\hat{\sigma}_{A+B \rightarrow c\bar{c}[^3S_1] + X(P_T)}$ is the perturbative cross section for $c\bar{c}$ pair production in color singlet and spin triplet state and the non-perturbative matrix element $< 0 | \mathcal{O}_{j/\psi} | 0 >$ represents the probability of $c\bar{c}$ pair in color singlet and spin triplet state to produce j/ψ . In eq. (6) the notation 3S_1 stands for the spectral notation $^{2S+1}L_J$ of the j/ψ where $S = 1, L = 0, J = 1$ are the spin, orbital angular momentum and total angular momentum respectively.

The definition of the non-perturbative matrix element $< 0 | \mathcal{O}_{j/\psi} | 0 >$ can be found as follows. The amplitude for the $c\bar{c}$ to form the specific hadron H plus other hadrons X is given by

$$< H + X | c\bar{c} > . \quad (7)$$

The amplitude in eq. (7) gives the probability of the hadron H production from the $c\bar{c}$ by using the non-perturbative matrix element

$$\begin{aligned} < 0 | \mathcal{O}_H | 0 > &= \sum_X < c\bar{c} | H + X > < H + X | c\bar{c} > = \sum_X < c\bar{c} | a_H^\dagger | X > < X | a_H | c\bar{c} > \\ &= < c\bar{c} | a_H^\dagger a_H | c\bar{c} > \end{aligned} \quad (8)$$

where a_H^\dagger is the creation operator of the hadron. Writing the Dirac wave function of the quark in terms of creation and annihilation operators a^\dagger, a and then using the definition of the charm quark state $|c>$ formed from the QCD vacuum $|0>$ by using the creation operator a^\dagger :

$$|c> = a^\dagger |0> \quad (9)$$

we find from eq. (8) that the definition of the non-perturbative matrix element at the origin for the η_c production from color singlet $c\bar{c}$ pair is given by

$$< 0 | \mathcal{O}_H | 0 > = < 0 | \chi^\dagger \psi a_H^\dagger a_H \psi^\dagger \chi | 0 > \quad (10)$$

where ψ, χ are two component Dirac spinors of the heavy quark wave function. Similar to the derivation of eq. (10) we find that the definition of the non-perturbative matrix element at the origin for the j/ψ production from $c\bar{c}$ pair in color singlet and spin triplet state is given by

$$\langle 0|\mathcal{O}_H|0\rangle = \langle 0|\chi^\dagger\sigma^i\psi a_H^\dagger a_H\psi^\dagger\sigma^i\chi|0\rangle \quad (11)$$

where σ^i is the Pauli spin matrix.

The wave function at the origin $R(0)$ for the j/ψ production from $c\bar{c}$ pair in color singlet mechanism is given by

$$|R(0)|^2 = \frac{2\pi}{N_c} \langle 0|\chi^\dagger\sigma^i\psi a_H^\dagger a_H\psi^\dagger\sigma^i\chi|0\rangle. \quad (12)$$

Note that in the perturbative cross section $d\hat{\sigma}_{A+B\rightarrow c\bar{c}[{}^3S_1]+X(P_T)}$ in eq. (6) the infrared divergences can occur due to the interaction of the c and/or \bar{c} with nearby light quark or gluon. This infrared divergences occur due to the soft (infrared) gluon interaction between the c and/or \bar{c} with nearby light quark or gluon. Hence it is necessary to prove that any non-canceling infrared divergences in the perturbative cross section $d\hat{\sigma}_{A+B\rightarrow c\bar{c}[{}^3S_1]+X(P_T)}$ in eq. (6) cancels with the corresponding infrared divergences in the non-perturbative matrix element $\langle 0|\mathcal{O}_{j/\psi}|0\rangle$ in eq. (6). If such a cancelation does not happen then one will predict infinite cross section $d\sigma_{A+B\rightarrow j/\psi+X(P_T)}$ for j/ψ production from eq. (6). Such cancelation of infrared divergences is called factorization. Hence it is necessary to prove factorization theorem of j/ψ production at high energy colliders.

Similar to eq. (9) in vacuum, using the definition of the $|c\rangle$ state in non-equilibrium by using the creation operator a^\dagger :

$$|c\rangle = a^\dagger|in\rangle \quad (13)$$

we find from eq. (8) that the non-perturbative matrix element for η_c production from color singlet $c\bar{c}$ pair in non-equilibrium QCD is given by

$$\langle in|\mathcal{O}_H|in\rangle = \langle in|\chi^\dagger\psi a_H^\dagger a_H\psi^\dagger\chi|in\rangle \quad (14)$$

where $|in\rangle$ is the initial state of the non-equilibrium QCD medium. Eq. (14) is similar to eq. (10) except that the vacuum expectation is replaced by medium average. Similar to the

derivation of eq. (14) we find that the non-perturbative matrix element for j/ψ production from $c\bar{c}$ pair in color singlet and spin triplet state in non-equilibrium QCD is given by

$$\langle in|\mathcal{O}_H|in \rangle = \langle in|\chi^\dagger \sigma^i \psi a_H^\dagger a_H \psi^\dagger \sigma^i \chi|in \rangle \quad (15)$$

which is similar to eq. (11) except that the vacuum expectation is replaced by medium average.

The infrared divergences can occur due to the (soft) gluon exchange between the c and/or \bar{c} with nearby light quark or gluon in non-equilibrium QCD medium at RHIC and LHC. Hence it is necessary to prove that any non-canceling infrared divergences due to these soft gluons exchange in non-equilibrium QCD medium cancel with the corresponding infrared divergences in the non-perturbative matrix element $\langle in|\mathcal{O}_{j/\psi}|in \rangle$. If such a cancelation does not happen then one will predict infinite cross section for j/ψ production at RHIC and LHC. Such cancelation of infrared divergences is called factorization in non-equilibrium QCD. Hence it is necessary to prove factorization theorem of j/ψ production in non-equilibrium QCD at RHIC and LHC.

In this paper we will prove that the infrared divergences due to the soft gluons exchange between charm quark and the nearby light-like quark (or gluon) in non-equilibrium QCD at RHIC and LHC exactly cancel with the corresponding infrared divergences due to the soft gluons exchange between anticharm quark and the same nearby light-like quark (or gluon) at all order in coupling constant in the j/ψ production from color singlet $c\bar{c}$ pair in non-equilibrium QCD. This proves the factorization theorem of j/ψ production in non-equilibrium QCD at RHIC and LHC at all order in coupling constant.

IV. INFRARED DIVERGENCE AND UNPHYSICAL LONGITUDINAL POLARIZATION OF THE GAUGE FIELD IN QUANTUM FIELD THEORY

Proof of factorization theorem is usually given in diagrammatic method by using pQCD [20–24]. However, since the matrix element $\langle 0|\chi^\dagger \sigma^i \psi a_H^\dagger a_H \psi^\dagger \sigma^i \chi|0 \rangle$ for j/ψ production from $c\bar{c}$ pair in eq. (11) is the non-perturbative matrix element it is useful to use the path integral formulation of QCD to prove factorization theorem. In fact, the proof of factorization theorem using path integral formulation is enormously simplified [28] in comparison to diagrammatic method. Also path integral method naturally proves factorization theorem

at all order in coupling constant whereas the diagrammatic method using pQCD suffers difficulties beyond certain order of coupling constant calculation [24].

Below we will briefly describe the general technique used in the path integral formulation in any quantum field theory to prove factorization theorem. Since we are interested in the soft-gluons exchange between the charm (and/or anticharm) quark with the nearby light-like quark and/or gluon we need to study the infrared behavior of the non-perturbative correlation function of the type $\langle 0 | \bar{\Psi}(x_1) \Psi(x_1) \bar{\Psi}(x_2) \Psi(x_2) | 0 \rangle$ in QCD in the presence of light-like Wilson line, in order to study j/ψ production from color singlet $c\bar{c}$ pair. Since a light-like charge produces pure gauge field in classical mechanics [20, 32, 33] and in quantum field theory (see the next section), one can study the infrared behavior of the non-perturbative correlation function of the type $\langle 0 | \bar{\Psi}(x_1) \Psi(x_1) \bar{\Psi}(x_2) \Psi(x_2) | 0 \rangle$ in QCD by using the path integral formulation of the background field method of QCD in the presence of pure gauge background field in order to prove factorization infrared divergences at all order in coupling constant [28].

Note that the pure gauge field in quantum field theory corresponds to longitudinal polarization of the gauge field. In this section we will show how the longitudinal polarization of the gauge field is used in quantum field theory to describe soft (infrared) divergences. Since the argument is valid in any quantum field theory let us see how the pure gauge field is used in QED to prove factorization of infrared divergences in QED. Consider an incoming electron of four momentum p^μ and mass m emitting a real photon of four momentum k^μ in QED. The corresponding Feynman diagram contribution is given by [34]

$$\mathcal{M} = \frac{1}{\gamma_\nu p^\nu - \gamma_\nu k^\nu - m} \gamma_\mu \epsilon^\mu(k) u(p) = -\frac{p \cdot \epsilon(k)}{p \cdot k} u(p) + \frac{k^\nu \gamma_\nu \gamma_\mu \epsilon^\mu(k)}{2p \cdot k} u(p) \quad (16)$$

where we write

$$\mathcal{M}_{\text{eikonal}} = -\frac{p \cdot \epsilon(k)}{p \cdot k} u(p) \quad (17)$$

and

$$\mathcal{M}_{\text{non-eikonal}} = \frac{k^\nu \gamma_\nu \gamma_\mu \epsilon^\mu(k)}{2p \cdot k} u(p). \quad (18)$$

From eq. (3.2) of [34] we write the gauge field as

$$\epsilon^\mu(k) = \left[\epsilon^\mu(k) - k^\mu \frac{p \cdot \epsilon(k)}{p \cdot k} \right] + k^\mu \frac{p \cdot \epsilon(k)}{p \cdot k} = \epsilon_{\text{phys}}^\mu(k) + \epsilon_{\text{pure}}^\mu(k) \quad (19)$$

where

$$\epsilon_{\text{phys}}^\mu(k) = [\epsilon^\mu(k) - k^\mu \frac{p \cdot \epsilon(k)}{p \cdot k}] \quad (20)$$

is the physical gauge field [corresponding to transverse polarization of the gauge field] and

$$\epsilon_{\text{pure}}^\mu(k) = k^\mu \frac{p \cdot \epsilon(k)}{p \cdot k} \quad (21)$$

is the pure gauge field [corresponding to longitudinal polarization of the gauge field].

Now using eq. (19) in (16) we find that the total contribution of the Feynman diagram is given by

$$\mathcal{M} = \mathcal{M}_{\text{eikonal}} + \mathcal{M}_{\text{non-eikonal}} \quad (22)$$

where

$$\mathcal{M}_{\text{eikonal}} = -\frac{p \cdot \epsilon_{\text{phys}}(k)}{p \cdot k} u(p) - \frac{p \cdot \epsilon_{\text{pure}}(k)}{p \cdot k} u(p) = -\frac{p \cdot \epsilon_{\text{pure}}(k)}{p \cdot k} u(p) \quad (23)$$

and

$$\mathcal{M}_{\text{non-eikonal}} = \frac{k^\nu \gamma_\nu \gamma_\mu \epsilon_{\text{phys}}^\mu(k)}{2p \cdot k} u(p) + \frac{k^\nu \gamma_\nu \gamma_\mu \epsilon_{\text{pure}}^\mu(k)}{2p \cdot k} u(p) = \frac{k^\nu \gamma_\nu \gamma_\mu \epsilon_{\text{phys}}^\mu(k)}{2p \cdot k} u(p). \quad (24)$$

Hence in the soft photon limit $(k_0, k_1, k_2, k_3) \rightarrow 0$ we find from the eqs. (16) and (23) that

$$-\mathcal{M}_{\text{eikonal}} = \frac{p \cdot \epsilon(k)}{p \cdot k} u(p) = \frac{p \cdot \epsilon_{\text{pure}}(k)}{p \cdot k} u(p) \rightarrow \infty \quad \text{as} \quad (k_0, k_1, k_2, k_3) \rightarrow 0 \quad (25)$$

which implies that the physical gauge field [corresponding to transverse polarization] does not contribute to the soft (infrared) divergences in quantum field theory and the soft (infrared) divergences can be calculated by using pure gauge field [corresponding to longitudinal polarization] in quantum field theory.

Similarly from eqs. (16) and (24) we find that

$$\mathcal{M}_{\text{non-eikonal}} = \frac{k^\nu \gamma_\nu \gamma_\mu \epsilon^\mu(k)}{2p \cdot k} u(p) = \frac{k^\nu \gamma_\nu \gamma_\mu \epsilon_{\text{phys}}^\mu(k)}{2p \cdot k} u(p) \rightarrow \text{finite} \quad \text{as} \quad (k_0, k_1, k_2, k_3) \rightarrow 0 \quad (26)$$

which contribute to the finite part of the cross section which implies that pure gauge field [corresponding to longitudinal polarization] does not contribute to the finite cross section and the finite cross section can be calculated by using physical gauge field [corresponding to transverse polarization].

Hence we find that the non-eikonal-line part of the diagram as given by eq. (26) is necessary if we are calculating the finite value of the cross section but it is not necessary if we are calculating the relevant infrared divergence behavior. The relevant infrared divergence behavior can be calculated by using the eikonal-line part of the diagram as given by eq. (25).

Hence we find that we do not need to calculate the finite value of the cross section (or the full cross section) [which will require the non-eikonal-line part of the diagram as given by eq. (26)] to study the relevant infrared divergence behavior. The relevant infrared divergence behavior can be calculated by using eikonal approximation as given by eq. (25).

From eq. (26) we find that

$$\mathcal{M}_{\text{non-eikonal}}^{\text{pure gauge field}} = \frac{k^\nu \gamma_\nu \gamma_\mu \epsilon_\mu^{\text{pure}}(k)}{2p \cdot k} u(p) = 0. \quad (27)$$

We are interested in the infrared divergence behavior due to the presence of the light-like Wilson line. We will show in the next section that the eikonal current of the light-like charge generates pure gauge field in quantum field theory. Hence from eqs. (25) and (27) we find that the soft (infrared) divergence behavior due to the presence light-like Wilson line can be studied by using pure gauge field in quantum field theory without modifying the finite value of the cross section

V. GAUGE FIELD GENERATED BY EIKONAL CURRENT OF LIGHT-LIKE WILSON LINE IN QUANTUM FIELD THEORY

In order to study factorization of infrared divergences by using the background field method of QED, the soft photon cloud traversed by the electron is represented by the pure gauge background field [25] due to the presence of the light-like Wilson line. As mentioned above, in classical mechanics the assertion that the gauge field that is produced by a highly relativistic (light-like) charge is a pure gauge field at all time-space position x^μ except at the position transverse to the motion of the charge at the time of closest approach [20, 32, 33]. One may ask a question if this assertion is correct in quantum field theory. In this section we will show that this assertion is correct in quantum field theory. We will use path integral formulation of the quantum field theory to show this.

The generating functional for the gauge field in the quantum field theory in the presence

of external source $J^\mu(x)$ in the path integral formulation is given by

$$Z[J] = \int [dA] e^{i \int d^4x [-\frac{1}{4} F_{\mu\nu}^2[A] - \frac{1}{2\alpha} (\partial_\mu A^\mu)^2 + J \cdot A]} \quad (28)$$

where

$$F^{\mu\nu}[A] = \partial^\mu A^\nu(x) - \partial^\nu A^\mu(x), \quad F_{\mu\nu}^2[A] = F^{\mu\nu}[A] F_{\mu\nu}[A]. \quad (29)$$

The effective action $S_{eff}[J]$ is given by [35]

$$\langle 0|0 \rangle_J = \frac{Z[J]}{Z[0]} = e^{iS_{eff}[J]} \quad (30)$$

where

$$S_{eff}[J] = -\frac{1}{2} \int d^4x d^4x' J^\mu(x) D_{\mu\nu}(x-x') J^\nu(x') \quad (31)$$

where $D_{\mu\nu}(x-x')$ is the photon propagator.

The photon propagator in the coordinate space is given by

$$D_{\mu\nu}(x-x') = \frac{1}{\partial^2} [g_{\mu\nu} + \frac{(\alpha-1)}{\partial^2} \partial_\mu \partial_\nu] \delta^{(4)}(x-x'). \quad (32)$$

Using eq. (32) in (31) we find

$$S_{eff}[J] = -\frac{1}{2} \int d^4x J^\mu(x) \frac{1}{\partial^2} [g_{\mu\nu} + \frac{(\alpha-1)}{\partial^2} \partial_\mu \partial_\nu] J^\nu(x). \quad (33)$$

From the continuity equation we have

$$\partial_\mu J^\mu(x) = 0. \quad (34)$$

Using eq. (34) in (33) we find

$$S_{eff}[J] = -\frac{1}{2} \int d^4x J^\mu(x) \frac{1}{\partial^2} J_\mu(x). \quad (35)$$

First of all, by using the path integral formulation of the quantum field theory we will derive Coulomb's law for static charge. Note that the derivation of the Coulomb's law by using path integral formulation of the quantum field theory is not necessary to prove factorization theorem. We have included it here only to demonstrate the correctness of the prediction of the path integral formulation in quantum field theory which we will use (see below) to show that the eikonal current of the light-like charge generates pure gauge field in quantum field theory.

In order to derive Coulomb's law by using path integral formulation of the quantum field theory let us consider the static charge at the position \vec{X} . The current density for this static charge is given by

$$J^\mu(x) = e\delta^{\mu 0}\delta^{(3)}(\vec{x} - \vec{X}). \quad (36)$$

Using eq. (36) in (35) we find

$$S_{eff}[J] = \frac{e^2}{2} \int d^4x \delta^{(3)}(\vec{x} - \vec{X}) \frac{1}{\nabla^2} \delta^{(3)}(\vec{x} - \vec{X}) = -\frac{e^2}{2} \int d^4x [\nabla \frac{1}{|\vec{x} - \vec{X}|}] \cdot \nabla \frac{1}{|\vec{x} - \vec{X}|} \quad (37)$$

which gives the effective lagrangian density

$$\mathcal{L}_{eff}(x) = -\frac{e^2}{2} [\nabla \frac{1}{|\vec{x} - \vec{X}|}] \cdot \nabla \frac{1}{|\vec{x} - \vec{X}|} = -\frac{E^2(x)}{2} \quad (38)$$

which reproduces the Coulomb's law. Hence we have shown that the assertion that a charge at rest generates a Coulomb gauge field is correct in quantum field theory.

Similarly using the above procedure in quantum field theory we will show that the assertion that the eikonal current of the light-like charge generates pure gauge field is correct in quantum field theory. This can be shown as follows.

In QED the infrared (or soft) divergence arises only from the emission of a photon for which all components of the four-momentum are small. As described in the previous section, the Eikonal propagator times the Eikonal vertex for a soft photon with momentum k interacting with a light-like electron moving with four momentum p^μ is given by [20, 23, 25, 26, 34, 36–42]

$$e \frac{p^\mu}{p \cdot k + i\epsilon} = e \frac{l^\mu}{l \cdot k + i\epsilon} \quad (39)$$

where l^μ is the four-velocity of the light-like electron. Note that when we say the "light-like electron" we mean the electron that is traveling at its highest speed which is arbitrarily close to the speed of light ($|\vec{l}| \sim 1$) as it can not travel exactly at speed of light ($|\vec{l}| = 1$) because it has finite mass even if the mass of the electron is very small. From eq. (39) we find

$$e \int \frac{d^4k}{(2\pi)^4} \frac{l \cdot A(k)}{l \cdot k + i\epsilon} = -ei \int_0^\infty d\lambda \int \frac{d^4k}{(2\pi)^4} e^{i\lambda k \cdot l} l \cdot A(k) = ie \int_0^\infty d\lambda l \cdot A(l\lambda) \quad (40)$$

where the photon field $A^\mu(x)$ and its Fourier transform $A^\mu(k)$ are related by

$$A^\mu(x) = \int \frac{d^4k}{(2\pi)^4} A^\mu(k) e^{ik \cdot x}. \quad (41)$$

From eq. (40) we find

$$ie \int_0^\infty d\lambda l \cdot A(l\lambda) = i \int d^4x J^\mu(x) A_\mu(x) \quad (42)$$

where the eikonal current density $J^\mu(x)$ of the light-like charge e is given by

$$J^\mu(x) = el^\mu \int d\lambda \delta^{(4)}(x - l\lambda), \quad l^2 = l^\mu l_\mu = 0. \quad (43)$$

Hence by using the path integral formulation of the quantum field theory we find by using eq. (43) in (35) that the effective action is given by

$$\begin{aligned} S_{eff}[J] &= -l^2 \frac{e^2}{2} \int d^4x \int d\lambda \delta^{(4)}(x - l\lambda) \frac{1}{\partial^2} \int d\lambda' \delta^{(4)}(x - l\lambda') \\ &= l^2 \frac{e^2}{2} \int d^4x \left[\partial^\mu \frac{1}{l \cdot (x - l \frac{x^2}{2l \cdot x})} \right] \left[\partial_\mu \frac{1}{l \cdot (x - l \frac{x^2}{2l \cdot x})} \right] \end{aligned} \quad (44)$$

which gives the effective lagrangian density

$$\mathcal{L}_{eff}(x) = l^2 \frac{e^2}{2} \frac{[l^\mu - \frac{l^2}{l \cdot x} (x^\mu - l^\mu \frac{x^2}{2l \cdot x})][l_\mu - \frac{l^2}{l \cdot x} (x_\mu - l_\mu \frac{x^2}{2l \cdot x})]}{[l \cdot x - l^2 \frac{x^2}{2l \cdot x}]^4}. \quad (45)$$

From eq. (45) we find the effective lagrangian density

$$\mathcal{L}_{eff}(x) = \frac{e^2}{2} \frac{(l^2)^2}{(l \cdot x)^4}, \quad \text{for} \quad l \cdot x \neq 0 \quad (46)$$

which gives

$$\mathcal{L}_{eff}(x) = 0 \quad (47)$$

at all time-space position x^μ except at the position transverse to the motion of the charge ($\vec{l} \cdot \vec{x} = 0$) at the time of closest approach ($x_0 = 0$) where we have used

$$l^2 = l^\mu l_\mu = 0 \quad (48)$$

where $l^\mu = \frac{v_c^\mu}{c}$ with $v_c^\mu = (c, \vec{v}_c)$ where $\vec{v}_c^2 = c^2$ and c is the speed of light. In this paper we have used natural unit where $c = 1$.

Hence from eqs. (30), (35), (47), (43) and (45) we find that the eikonal current for light-like charge generates pure gauge field in quantum field theory.

VI. PROOF OF FACTORIZATION OF J/ψ PRODUCTION AT HIGH ENERGY COLLIDERS

Arguments about the proof of factorization of j/ψ production in color singlet mechanism at high energy colliders by using pQCD diagrammatic method in QCD in vacuum is given in [30]. In this section we will prove the factorization theorem of j/ψ production from color singlet and spin triplet $c\bar{c}$ pair at high energy colliders at all order in coupling constant by using path integral formulation of QCD in vacuum. We will extend this path integral technique to prove factorization theorem of j/ψ production from color singlet and spin triplet $c\bar{c}$ pair in non-equilibrium QCD at RHIC and LHC at all order in coupling constant in section VII.

The generating functional in QCD including the heavy quark is given by [43, 44]

$$Z[J, \eta_u, \bar{\eta}_u, \eta_d, \bar{\eta}_d, \eta_s, \bar{\eta}_s, \eta_h, \bar{\eta}_h] = \int [dQ][d\bar{\psi}_1][d\psi_1][d\bar{\psi}_2][d\psi_2][d\bar{\psi}_3][d\psi_3][d\bar{\Psi}][d\Psi] \det\left(\frac{\delta(\partial_\mu Q^{\mu a})}{\delta\omega^b}\right) e^{i \int d^4x \left[-\frac{1}{4}F_{\mu\nu}^2[Q] - \frac{1}{2\alpha}(\partial_\mu Q^{\mu a})^2 + J \cdot Q + \sum_{l=1}^3 [\bar{\psi}_l[i\gamma^\mu \partial_\mu - m_l + gT^a \gamma^\mu Q_\mu^a] \psi_l + \bar{\eta}_l \psi_l + \bar{\psi}_l \eta_l] + \bar{\Psi}[i\gamma^\mu \partial_\mu - M + gT^a \gamma^\mu Q_\mu^a] \Psi + \bar{\eta}_h \Psi + \bar{\Psi} \eta_h \right]} \quad (49)$$

where $Q^{\mu a}$ is the quantum gluon field, the symbols $l = 1, 2, 3 = u, d, s$ stand for three light quarks u, d, s and the symbol h stands for heavy quark and

$$F_{\mu\nu}^a[Q] = \partial_\mu Q_\nu^a(x) - \partial_\nu Q_\mu^a(x) + gf^{abc}Q_\mu^b(x)Q_\nu^c(x), \quad F_{\mu\nu}^{a2}[Q] = F_{\mu\nu}^a[Q]F^{\mu\nu a}[Q]. \quad (50)$$

Note that the determinant $\det(\frac{\delta(\partial_\mu Q^{\mu a})}{\delta\omega^b})$ in eq. (49) can be expressed in terms of path integration over the ghost fields [44]. However, we will directly work with the determinant $\det(\frac{\delta(\partial_\mu Q^{\mu a})}{\delta\omega^b})$ in eq. (49).

For the heavy quark Dirac field $\Psi(x)$, the non-perturbative matrix element of the type $\langle 0 | \bar{\Psi}(x_1) \Psi(x_1) \bar{\Psi}(x_2) \Psi(x_2) | 0 \rangle$ in QCD is given by [25]

$$\begin{aligned} & \langle 0 | \bar{\Psi}(x_1) \Psi(x_1) \bar{\Psi}(x_2) \Psi(x_2) | 0 \rangle \\ &= \int [dQ][d\bar{\psi}_1][d\psi_1][d\bar{\psi}_2][d\psi_2][d\bar{\psi}_3][d\psi_3][d\bar{\Psi}][d\Psi] \bar{\Psi}(x_1) \Psi(x_1) \bar{\Psi}(x_2) \Psi(x_2) \det\left(\frac{\delta(\partial_\mu Q^{\mu a})}{\delta\omega^b}\right) \\ & e^{i \int d^4x \left[-\frac{1}{4}F_{\mu\nu}^2[Q] - \frac{1}{2\alpha}(\partial_\mu Q^{\mu a})^2 + \sum_{l=1}^3 [\bar{\psi}_l[i\gamma^\mu \partial_\mu - m_l + gT^a \gamma^\mu Q_\mu^a] \psi_l] + \bar{\Psi}[i\gamma^\mu \partial_\mu - M + gT^a \gamma^\mu Q_\mu^a] \Psi \right]} \end{aligned} \quad (51)$$

where the suppression of the normalization factor $Z[0]$ is understood as it will cancel in the final result (see eq. (73)).

As mentioned in sections IV and V the infrared divergences can be studied by using eikonal approximation in quantum field theory. We are interested in the infrared divergence behavior of the non-perturbative heavy quark-antiquark correlation function of the type $\langle 0 | \bar{\Psi}(x_1) \Psi(x_1) \bar{\Psi}(x_2) \Psi(x_2) | 0 \rangle$ in the presence of light-like Wilson line in QCD.

The Eikonal propagator times the Eikonal vertex for a gluon with four momentum k^μ interacting with a light-like quark moving with four velocity l^μ is given by [20, 24–27, 36–42]

$$gT^a \frac{l^\mu}{l \cdot k + i\epsilon}. \quad (52)$$

Hence for a light-like quark attached to infinite number of gluons we find the eikonal factor

$$\begin{aligned} & 1 + gT^a \int \frac{d^4k}{(2\pi)^4} \frac{l \cdot A^a(k)}{l \cdot k + i\epsilon} + g^2 \int \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} \frac{T^a l \cdot A^a(k_1) T^b l \cdot A^b(k_2)}{(l \cdot k_1 + i\epsilon)(l \cdot (k_1 + k_2) + i\epsilon)} + \dots \\ & = 1 + igT^a \int_0^\infty d\lambda l \cdot A^a(l\lambda) + g^2 i^2 \int_0^\infty d\lambda_1 \int_{\lambda_1}^\infty d\lambda_2 T^a l \cdot A^a(l\lambda_1) T^b l \cdot A^b(l\lambda_2) + \dots \\ & = 1 + igT^a \int_0^\infty d\lambda l \cdot A^a(l\lambda) + \frac{g^2 i^2}{2!} \mathcal{P} \int_0^\infty d\lambda_1 \int_0^\infty d\lambda_2 T^a l \cdot A^a(l\lambda_1) T^b l \cdot A^b(l\lambda_2) + \dots \\ & = \mathcal{P} \exp[ig \int_0^\infty d\lambda l \cdot A^a(l\lambda) T^a] \end{aligned} \quad (53)$$

which describes the infrared divergences arising from the infinite number of soft gluons exchange with the light-like quark where \mathcal{P} is the path ordering and the gluon field $A^{\mu a}(x)$ and its Fourier transform $A^{\mu a}(k)$ are related by

$$A^{\mu a}(x) = \int \frac{d^4k}{(2\pi)^4} A^{\mu a}(k) e^{ik \cdot x}. \quad (54)$$

. As mentioned in sections IV and V the light-like quark traveling with light-like four-velocity l^μ produces SU(3) pure gauge field at all the time-space position x^μ except at the position \vec{x} perpendicular to the direction of motion of the quark ($\vec{l} \cdot \vec{x} = 0$) at the time of closest approach [20, 32, 33]. When $A^{\mu a}(x) = A^{\mu a}(\lambda l)$ as in eq. (53) we find $\vec{l} \cdot \vec{x} = \lambda \vec{l} \cdot \vec{l} = \lambda \neq 0$ which implies that the light-like quark finds the gluon field $A^{\mu a}(x)$ in eq. (53) as the SU(3) pure gauge. The SU(3) pure gauge is given by

$$T^a A_\mu^a(x) = \frac{1}{ig} [\partial_\mu U(x)] U^{-1}(x), \quad U(x) = e^{igT^a \omega^a(x)} \quad (55)$$

which gives

$$U(x_f) = \mathcal{P} e^{ig \int_{x_i}^{x_f} dx^\mu A_\mu^a(x) T^a} U(x_i) = e^{igT^a \omega^a(x_f)}. \quad (56)$$

Hence when $A^{\mu a}(x) = A^{\mu a}(\lambda l)$ as in eq. (53) we find from eq. (56) that the light-like Wilson line in QCD for infrared divergences is given by [28]

$$\Phi(x) = \mathcal{P} e^{-ig \int_0^\infty d\lambda \cdot A^a(x+l\lambda) T^a} = e^{ig T^a \omega^a(x)}. \quad (57)$$

Hence the effect of infrared gluons interaction between the partons and the light-like Wilson line in QCD can be studied by putting the partons in the SU(3) pure gauge background field which implies that the infrared behavior of the non-perturbative correlation function of the type $\langle \bar{\Psi}(x)\Psi(x')\bar{\Psi}(x'')\Psi(x''')\dots \rangle$ in QCD due to the presence of light-like Wilson line in QCD can be studied by using the path integral method of the QCD in the presence of SU(3) pure gauge background field as given by eq. (55).

Background field method of QCD was originally formulated by 't Hooft [45] and later extended by Klueberg-Stern and Zuber [46, 47] and by Abbott [43]. This is an elegant formalism which can be useful to construct gauge invariant non-perturbative green's functions in QCD. This formalism is also useful to study quark and gluon production from classical chromo field via Schwinger mechanism [35], to compute β function in QCD [48], to perform calculations in lattice gauge theories [49] and to study evolution of QCD coupling constant in the presence of chromofield [50].

The generating functional in the background field method of QCD is given by

$$\begin{aligned} Z[A, J, \eta_u, \bar{\eta}_u, \eta_d, \bar{\eta}_d, \eta_s, \bar{\eta}_s, \eta_h, \bar{\eta}_h] &= \int [dQ][d\bar{\psi}_1][d\psi_1][d\bar{\psi}_2][d\psi_2][d\bar{\psi}_3][d\psi_3][d\bar{\Psi}][d\Psi] \det\left(\frac{\delta G^a(Q)}{\delta \omega^b}\right) \\ &\exp\left[i \int d^4x \left[-\frac{1}{4} F_{\mu\nu}^2[A+Q] - \frac{1}{2\alpha} (G^a(Q))^2 + J \cdot Q \right.\right. \\ &+ \sum_{l=1}^3 \left[\bar{\psi}_l [i\gamma^\mu \partial_\mu - m_l + gT^a \gamma^\mu (A+Q)_\mu^a] \psi_l + \bar{\eta}_l \psi_l + \bar{\psi}_l \eta_l \right] \\ &\left. + \bar{\Psi} [i\gamma^\mu \partial_\mu - M + gT^a \gamma^\mu (A+Q)_\mu^a] \Psi + \bar{\eta}_h \Psi + \bar{\Psi} \eta_h \right] \end{aligned} \quad (58)$$

where the gauge fixing term is given by

$$G^a(Q) = \partial_\mu Q^{\mu a} + g f^{abc} A_\mu^b Q^{\mu c} = D_\mu[A] Q^{\mu a} \quad (59)$$

which depends on the background field $A^{\mu a}(x)$ and

$$F_{\mu\nu}^a[A+Q] = \partial_\mu [A_\nu^a + Q_\nu^a] - \partial_\nu [A_\mu^a + Q_\mu^a] + g f^{abc} [A_\mu^b + Q_\mu^b] [A_\nu^c + Q_\nu^c]. \quad (60)$$

We have followed the notations of [43, 45, 46] and accordingly we have denoted the quantum gluon field by $Q^{\mu a}$ and the background field by $A^{\mu a}$. The gauge fixing term $\frac{1}{2\alpha} (G^a(Q))^2$ in

eq. (58) [where $G^a(Q)$ is given by eq. (59)] is invariant for gauge transformation of A_μ^a :

$$\delta A_\mu^a = g f^{abc} A_\mu^b \omega^c + \partial_\mu \omega^a, \quad (\text{type I transformation}) \quad (61)$$

provided one also performs a homogeneous transformation of Q_μ^a [43, 46]:

$$\delta Q_\mu^a = g f^{abc} Q_\mu^b \omega^c. \quad (62)$$

The gauge transformation of background field A_μ^a as given by eq. (61) along with the homogeneous transformation of Q_μ^a in eq. (62) gives

$$\delta(A_\mu^a + Q_\mu^a) = g f^{abc} (A_\mu^b + Q_\mu^b) \omega^c + \partial_\mu \omega^a \quad (63)$$

which leaves $-\frac{1}{4}F_{\mu\nu}^2[A+Q]$ invariant in eq. (58).

For the heavy quark Dirac field $\Psi(x)$, the non-perturbative matrix element of the type $\langle 0 | \bar{\Psi}(x_1) \Psi(x_1) \bar{\Psi}(x_2) \Psi(x_2) | 0 \rangle$ in the background field method of QCD is given by [25]

$$\begin{aligned} & \langle 0 | \bar{\Psi}(x_1) \Psi(x_1) \bar{\Psi}(x_2) \Psi(x_2) | 0 \rangle_A \\ &= \int [dQ][d\bar{\psi}_1][d\psi_1][d\bar{\psi}_2][d\psi_2][d\bar{\psi}_3][d\psi_3][d\bar{\Psi}][d\Psi] \bar{\Psi}(x_1) \Psi(x_1) \bar{\Psi}(x_2) \Psi(x_2) \det\left(\frac{\delta G^a(Q)}{\delta \omega^b}\right) \\ & \exp\left[i \int d^4x \left[-\frac{1}{4}F_{\mu\nu}^2[A+Q] - \frac{1}{2\alpha}(G^a(Q))^2 + \sum_{l=1}^3 \left[\bar{\psi}_l[i\gamma^\mu \partial_\mu - m_l + gT^a \gamma^\mu (A+Q)_\mu^a] \psi_l\right] \right.\right. \\ & \left. \left. + \bar{\Psi}[i\gamma^\mu \partial_\mu - M + gT^a \gamma^\mu (A+Q)_\mu^a] \Psi\right]\right] \end{aligned} \quad (64)$$

where the suppression of the normalization factor $Z[0]$ is understood as it will cancel in the final result (see eq. (73)). By changing $Q \rightarrow Q - A$ in eq. (64) we find that

$$\begin{aligned} & \langle 0 | \bar{\Psi}(x_1) \Psi(x_1) \bar{\Psi}(x_2) \Psi(x_2) | 0 \rangle_A \\ &= \int [dQ][d\bar{\psi}_1][d\psi_1][d\bar{\psi}_2][d\psi_2][d\bar{\psi}_3][d\psi_3][d\bar{\Psi}][d\Psi] \bar{\Psi}(x_1) \Psi(x_1) \bar{\Psi}(x_2) \Psi(x_2) \det\left(\frac{\delta G_f^a(Q)}{\delta \omega^b}\right) \\ & \exp\left[i \int d^4x \left[-\frac{1}{4}F_{\mu\nu}^2[Q] - \frac{1}{2\alpha}(G_f^a(Q))^2 + \sum_{l=1}^3 \left[\bar{\psi}_l[i\gamma^\mu \partial_\mu - m_l + gT^a \gamma^\mu Q_\mu^a] \psi_l\right] \right.\right. \\ & \left. \left. + \bar{\Psi}[i\gamma^\mu \partial_\mu - M + gT^a \gamma^\mu Q_\mu^a] \Psi\right]\right] \end{aligned} \quad (65)$$

where the gauge fixing term from eq. (59) becomes

$$G_f^a(Q) = \partial_\mu Q^{\mu a} + g f^{abc} A_\mu^b Q^{\mu c} - \partial_\mu A^{\mu a} = D_\mu[A] Q^{\mu a} - \partial_\mu A^{\mu a} \quad (66)$$

and from eq. (62) [by using eq. (61)] we find

$$\delta Q_\mu^a = -g f^{abc} \omega^b Q_\mu^c + \partial_\mu \omega^a. \quad (67)$$

For finite transformation eq. (67) becomes

$$T^a Q'_\mu(x) = U(x) T^a Q_\mu(x) U^{-1}(x) + \frac{1}{ig} [\partial_\mu U(x)] U^{-1}(x), \quad U(x) = e^{igT^a \omega^a(x)}. \quad (68)$$

The fermion fields transform as

$$\psi'_l(x) = e^{igT^a \omega^a(x)} \psi_l(x), \quad \Psi'(x) = e^{igT^a \omega^a(x)} \Psi(x). \quad (69)$$

Changing the variables of integration from unprimed to primed variables in eq. (65) we find

$$\begin{aligned} & < 0 | \bar{\Psi}(x_1) \Psi(x_1) \bar{\Psi}(x_2) \Psi(x_2) | 0 >_A \\ &= \int [dQ'] [d\bar{\psi}'_1] [d\psi'_1] [d\bar{\psi}'_2] [d\psi'_2] [d\bar{\psi}'_3] [d\psi'_3] [d\bar{\Psi}'] [d\Psi'] \bar{\Psi}'(x_1) \Psi'(x_1) \bar{\Psi}'(x_2) \Psi'(x_2) \det\left(\frac{\delta G_f^a(Q')}{\delta \omega^b}\right) \\ & \exp[i \int d^4x \{-\frac{1}{4} F_{\mu\nu}^2[Q'] - \frac{1}{2\alpha} (G_f^a(Q'))^2 + \sum_{l=1}^3 [\bar{\psi}'_l [i\gamma^\mu \partial_\mu - m_l + gT^a \gamma^\mu Q'_\mu] \psi'_l] \\ & + \bar{\Psi}' [i\gamma^\mu \partial_\mu - M + gT^a \gamma^\mu Q'_\mu] \Psi'] \end{aligned} \quad (70)$$

because the change of variables from unprimed to primed variables does not change the value of the integration.

From eqs. (68) and (69) we find [28]

$$\begin{aligned} [dQ'] &= [dQ], \quad [d\bar{\psi}'_1] [d\psi'_1] = [d\bar{\psi}_1] [d\psi_1], \quad [d\bar{\psi}'_2] [d\psi'_2] = [d\bar{\psi}_2] [d\psi_2], \quad [d\bar{\psi}'_3] [d\psi'_3] = [d\bar{\psi}_3] [d\psi_3], \\ [d\bar{\Psi}'] [d\Psi'] &= [d\bar{\Psi}] [d\Psi], \quad \bar{\psi}'_l [i\gamma^\mu \partial_\mu - m_l + gT^a \gamma^\mu Q'_\mu] \psi'_l = \bar{\psi}_l [i\gamma^\mu \partial_\mu - m_l + gT^a \gamma^\mu Q_\mu] \psi_l, \\ \bar{\Psi}' [i\gamma^\mu \partial_\mu - M + gT^a \gamma^\mu Q'_\mu] \Psi' &= \bar{\Psi} [i\gamma^\mu \partial_\mu - M + gT^a \gamma^\mu Q_\mu] \Psi, \quad F_{\mu\nu}^2[Q'] = F_{\mu\nu}^2[Q] \\ (G_f^a(Q'))^2 &= (\partial_\mu Q^{\mu a}(x))^2, \quad \det\left[\frac{\delta G_f^a(Q')}{\delta \omega^b}\right] = \det\left[\frac{\delta(\partial_\mu Q^{\mu a}(x))}{\delta \omega^b}\right]. \end{aligned} \quad (71)$$

Using eq. (71) in (70) we find

$$\begin{aligned} & < 0 | \bar{\Psi}(x_1) \Psi(x_1) \bar{\Psi}(x_2) \Psi(x_2) | 0 > \\ &= \int [dQ] [d\bar{\psi}_1] [d\psi_1] [d\bar{\psi}_2] [d\psi_2] [d\bar{\psi}_3] [d\psi_3] [d\bar{\Psi}] [d\Psi] \bar{\Psi}(x_1) \Psi(x_1) \bar{\Psi}(x_2) \Psi(x_2) \det\left(\frac{\delta(\partial_\mu Q^{\mu a})}{\delta \omega^b}\right) \\ & e^{i \int d^4x \{-\frac{1}{4} F_{\mu\nu}^2[Q] - \frac{1}{2\alpha} (\partial_\mu Q^{\mu a})^2 + \sum_{l=1}^3 [\bar{\psi}_l [i\gamma^\mu \partial_\mu - m_l + gT^a \gamma^\mu Q_\mu] \psi_l] + \bar{\Psi} [i\gamma^\mu \partial_\mu - M + gT^a \gamma^\mu Q_\mu] \Psi]}. \end{aligned} \quad (72)$$

From eqs. (72) and (51) we find

$$< 0 | \bar{\Psi}(x_1) \Psi(x_1) \bar{\Psi}(x_2) \Psi(x_2) | 0 > = < 0 | \bar{\Psi}(x_1) \Psi(x_1) \bar{\Psi}(x_2) \Psi(x_2) | 0 >_A \quad (73)$$

which proves factorization of infrared divergences at all order in coupling constant. Eq. (73) is valid in covariant gauge, in light-cone gauge, in general axial gauges, in general non-covariant gauges and in general Coulomb gauge etc. respectively [28].

From eq. (73) we find that the non-perturbative matrix element for j/ψ production from $c\bar{c}$ pair in color singlet and spin triplet state in eq. (11) as given by

$$\langle 0|\mathcal{O}_H|0\rangle = \langle 0|\chi^\dagger\sigma^i\psi a_H^\dagger a_H\psi^\dagger\sigma^i\chi|0\rangle \quad (74)$$

is consistent with factorization theorem of j/ψ production at high energy colliders at all order in coupling constant.

Eq. (73) proves the factorization theorem of j/ψ production at high energy colliders at all order in coupling constant where $\langle 0|\bar{\Psi}(x_1)\Psi(x_1)\bar{\Psi}(x_2)\Psi(x_2)|0\rangle$ is the heavy quark-antiquark gauge invariant non-perturbative correlation function in QCD and $\langle 0|\bar{\Psi}(x_1)\Psi(x_1)\bar{\Psi}(x_2)\Psi(x_2)|0\rangle_A$ is the corresponding heavy quark-antiquark gauge invariant non-perturbative correlation function in QCD in the presence of light-like Wilson line. From eq. (73) one finds that the infrared divergences due to the soft gluons exchange between charm quark and the nearby light-like quark (or gluon) at high energy colliders exactly cancel with the corresponding infrared divergences due to the soft gluons exchange between anticharm quark and the same nearby light-like quark (or gluon) at all order in coupling constant in the j/ψ production from color singlet $c\bar{c}$ pair at high energy colliders. This proves the factorization theorem of j/ψ production at high energy colliders at all order in coupling constant.

VII. PROOF OF FACTORIZATION OF J/ψ PRODUCTION IN NON-EQUILIBRIUM QCD AT RHIC AND LHC

The generating functional in non-equilibrium QCD (including heavy quark) in the path integral formulation is given by eq. (3). The nonequilibrium-nonperturbative heavy quark-antiquark correlation function of the type $\langle in|\bar{\Psi}_r(x_1)\Psi_r(x_1)\bar{\Psi}_s(x_2)\Psi_s(x_2)|in\rangle$ in QCD is given by eq. (5) which gives

$$\begin{aligned} & \langle in|\bar{\Psi}_r(x_1)\Psi_r(x_1)\bar{\Psi}_s(x_2)\Psi_s(x_2)|in\rangle = \\ & = \int [dQ_+][dQ_-][d\bar{\psi}_{1+}][d\bar{\psi}_{1-}][d\psi_{1+}][d\psi_{1-}][d\bar{\psi}_{2+}][d\bar{\psi}_{2-}][d\psi_{2+}][d\psi_{2-}][d\bar{\psi}_{3+}][d\bar{\psi}_{3-}][d\psi_{3+}][d\psi_{3-}] \\ & [d\bar{\Psi}_+][d\bar{\Psi}_-][d\Psi_+][d\Psi_-] \bar{\Psi}_r(x_1)\Psi_r(x_1)\bar{\Psi}_s(x_2)\Psi_s(x_2) \times \det\left(\frac{\delta\partial_\mu Q_+^{\mu a}}{\delta\omega_+^b}\right) \times \det\left(\frac{\delta\partial_\mu Q_-^{\mu a}}{\delta\omega_-^b}\right) \\ & \exp[i \int d^4x [-\frac{1}{4}(F_{\mu\nu}^{a2}[Q_+] - F_{\mu\nu}^{a2}[Q_-]) - \frac{1}{2\alpha}((\partial_\mu Q_+^{\mu a})^2 - (\partial_\mu Q_-^{\mu a})^2)] \end{aligned}$$

$$\begin{aligned}
& + \sum_{l=1}^3 \bar{\psi}_{l+} [i\gamma^\mu \partial_\mu - m_l + gT^a \gamma^\mu Q_{\mu+}^a] \psi_{l+} - \sum_{l=1}^3 \bar{\psi}_{l-} [i\gamma^\mu \partial_\mu - m_l + gT^a \gamma^\mu Q_{\mu-}^a] \psi_{l-} \\
& + \bar{\Psi}_+ [i\gamma^\mu \partial_\mu - M + gT^a \gamma^\mu Q_{\mu+}^a] \Psi_+ - \bar{\Psi}_- [i\gamma^\mu \partial_\mu - M + gT^a \gamma^\mu Q_{\mu-}^a] \Psi_-] \\
& \times \langle Q_+, \psi_+^u, \bar{\psi}_+^u, \psi_+^d, \bar{\psi}_+^d, \psi_+^s, \bar{\psi}_+^s, \Psi_+, \bar{\Psi}_+, 0 | \rho | 0, \Psi_-, \bar{\Psi}_-, \bar{\psi}_-^s, \psi_-^s, \bar{\psi}_-^d, \psi_-^d, \bar{\psi}_-^u, \psi_-^u, Q_- \rangle
\end{aligned} \tag{75}$$

where the repeated (closed-time path) indices $r, s = +, -$ are not summed and the suppression of the normalization factor $Z[0]$ is understood as it will cancel in the final result, see eq. (82).

The generating functional in the background field method of QCD in non-equilibrium QCD (including heavy quark) in the path integral formulation is given by [43, 45–47, 53, 54]

$$\begin{aligned}
& Z[\rho, A, J_+, J_-, \eta_+^u, \eta_-^u, \bar{\eta}_+^u, \bar{\eta}_-^u, \eta_+^d, \eta_-^d, \bar{\eta}_+^d, \bar{\eta}_-^d, \eta_+^s, \eta_-^s, \bar{\eta}_+^s, \bar{\eta}_-^s, \eta_+^h, \eta_-^h, \bar{\eta}_+^h, \bar{\eta}_-^h] \\
& = \int [dQ_+][dQ_-][d\bar{\psi}_{1+}][d\bar{\psi}_{1-}][d\psi_{1+}][d\psi_{1-}][d\bar{\psi}_{2+}][d\bar{\psi}_{2-}][d\psi_{2+}][d\psi_{2-}][d\bar{\psi}_{3+}][d\bar{\psi}_{3-}][d\psi_{3+}][d\psi_{3-}] \\
& [d\bar{\Psi}_+][d\bar{\Psi}_-][d\Psi_+][d\Psi_-] \times \det\left(\frac{\delta G^a(Q_+)}{\delta \omega_+^b}\right) \times \det\left(\frac{\delta G^a(Q_-)}{\delta \omega_-^b}\right) \times \exp[i \int d^4x [-\frac{1}{4}(F_{\mu\nu}^a)^2 [Q_+ + A_+] \\
& - F_{\mu\nu}^a [Q_- + A_-]) - \frac{1}{2\alpha}((G^a(Q_+))^2 - (G^a(Q_-))^2) + \sum_{l=1}^3 \bar{\psi}_{l+} [i\gamma^\mu \partial_\mu - m_l + gT^a \gamma^\mu (Q_+ + A_+)_\mu^a] \psi_{l+} \\
& - \sum_{l=1}^3 \bar{\psi}_{l-} [i\gamma^\mu \partial_\mu - m_l + gT^a \gamma^\mu (Q_- + A_-)_\mu^a] \psi_{l-} + \bar{\Psi}_+ [i\gamma^\mu \partial_\mu - M + gT^a \gamma^\mu (Q_+ + A_+)_\mu^a] \Psi_+ \\
& - \bar{\Psi}_- [i\gamma^\mu \partial_\mu - M + gT^a \gamma^\mu (Q_- + A_-)_\mu^a] \Psi_- + J_+ \cdot Q_+ - J_- \cdot Q_- \\
& + \sum_{l=1}^3 [\bar{\eta}_{l+} \cdot \psi_{l+} - \bar{\eta}_{l-} \cdot \psi_{l-} + \bar{\psi}_{l+} \cdot \eta_{l+} - \bar{\psi}_{l-} \cdot \eta_{l-}] + \bar{\eta}_{h+} \cdot \Psi_+ - \bar{\eta}_{h-} \cdot \Psi_- + \bar{\Psi}_+ \cdot \eta_{h+} - \bar{\Psi}_- \cdot \eta_{h-}] \\
& \times \langle Q_+ + A_+, \psi_+^u, \bar{\psi}_+^u, \psi_+^d, \bar{\psi}_+^d, \psi_+^s, \bar{\psi}_+^s, \Psi_+, \bar{\Psi}_+, 0 | \rho | 0, \Psi_-, \bar{\Psi}_-, \bar{\psi}_-^s, \psi_-^s, \bar{\psi}_-^d, \psi_-^d, \bar{\psi}_-^u, \psi_-^u, Q_- + A_- \rangle
\end{aligned} \tag{76}$$

where the gauge fixing term $G^a(Q)$ is given by eq. (59) which depends on the background field $A^{\mu a}(x)$, the $F_{\mu\nu}^a[Q + A]$ is given by eq. (60), the δQ_μ^a and δA_μ^a are given by eqs. (62) and (61) respectively.

The nonequilibrium-nonperturbative heavy quark-antiquark correlation function of the type $\langle in | \bar{\Psi}_r(x_1) \Psi_r(x_1) \bar{\Psi}_s(x_2) \Psi_s(x_2) | in \rangle$ in the background field method of QCD is given by

$$\begin{aligned}
& \langle in | \bar{\Psi}_r(x_1) \Psi_r(x_1) \bar{\Psi}_s(x_2) \Psi_s(x_2) | in \rangle_A \\
& = \int [dQ_+][dQ_-][d\bar{\psi}_{1+}][d\bar{\psi}_{1-}][d\psi_{1+}][d\psi_{1-}][d\bar{\psi}_{2+}][d\bar{\psi}_{2-}][d\psi_{2+}][d\psi_{2-}][d\bar{\psi}_{3+}][d\bar{\psi}_{3-}][d\psi_{3+}][d\psi_{3-}]
\end{aligned}$$

$$\begin{aligned}
& [d\bar{\Psi}_+][d\bar{\Psi}_-][d\Psi_+][d\Psi_-] \bar{\Psi}_r(x_1)\Psi_r(x_1)\bar{\Psi}_s(x_2)\Psi_s(x_2) \\
& \times \det\left(\frac{\delta G^a(Q_+)}{\delta\omega_+^b}\right) \times \det\left(\frac{\delta G^a(Q_-)}{\delta\omega_-^b}\right) \times \exp\left[i \int d^4x \left[-\frac{1}{4}(F_{\mu\nu}^{a2}[Q_+ + A_+] \right. \right. \\
& \left. \left. - F_{\mu\nu}^{a2}[Q_- + A_-] \right) - \frac{1}{2\alpha}((G^a(Q_+))^2 - (G^a(Q_-))^2) + \sum_{l=1}^3 \bar{\psi}_{l+}[i\gamma^\mu\partial_\mu - m_l + gT^a\gamma^\mu(Q_+ + A_+)_\mu^a]\psi_{l+} \right. \\
& \left. - \sum_{l=1}^3 \bar{\psi}_{l-}[i\gamma^\mu\partial_\mu - m_l + gT^a\gamma^\mu(Q_- + A_-)_\mu^a]\psi_{l-} + \bar{\Psi}_+[i\gamma^\mu\partial_\mu - M + gT^a\gamma^\mu(Q_+ + A_+)_\mu^a]\Psi_+ \right. \\
& \left. - \bar{\Psi}_-[i\gamma^\mu\partial_\mu - M + gT^a\gamma^\mu(Q_- + A_-)_\mu^a]\Psi_-] \right] \\
& \times \langle Q_+ + A_+, \psi_+^u, \bar{\psi}_+^u, \psi_+^d, \bar{\psi}_+^d, \psi_+^s, \bar{\psi}_+^s, \Psi_+, \bar{\Psi}_+, 0 | \rho | 0, \Psi_-, \bar{\Psi}_-, \bar{\psi}_-^s, \psi_-^s, \bar{\psi}_-^d, \psi_-^d, \bar{\psi}_-^u, \psi_-^u, Q_- + A_- \rangle
\end{aligned} \tag{77}$$

where the suppression of the normalization factor $Z[0]$ is understood as it will cancel in the final result, see eq. (82).

In order to study the infrared behavior of the nonequilibrium-nonperturbative heavy quark-antiquark correlation function we proceed as follows. By changing the integration variable $Q \rightarrow Q - A$ in the right hand side of eq. (77) we find

$$\begin{aligned}
& \langle in | \bar{\Psi}_r(x_1)\Psi_r(x_1)\bar{\Psi}_s(x_2)\Psi_s(x_2) | in \rangle_A \\
& = \int [dQ_+][dQ_-][d\bar{\psi}_{1+}][d\bar{\psi}_{1-}][d\psi_{1+}][d\psi_{1-}][d\bar{\psi}_{2+}][d\bar{\psi}_{2-}][d\psi_{2+}][d\psi_{2-}][d\bar{\psi}_{3+}][d\bar{\psi}_{3-}][d\psi_{3+}][d\psi_{3-}] \\
& [d\bar{\Psi}_+][d\bar{\Psi}_-][d\Psi_+][d\Psi_-] \bar{\Psi}_r(x_1)\Psi_r(x_1)\bar{\Psi}_s(x_2)\Psi_s(x_2) \\
& \times \det\left(\frac{\delta G_f^a(Q_+)}{\delta\omega_+^b}\right) \times \det\left(\frac{\delta G_f^a(Q_-)}{\delta\omega_-^b}\right) \times \exp\left[i \int d^4x \left[-\frac{1}{4}(F_{\mu\nu}^{a2}[Q_+] \right. \right. \\
& \left. \left. - F_{\mu\nu}^{a2}[Q_-] \right) - \frac{1}{2\alpha}((G_f^a(Q_+))^2 - (G_f^a(Q_-))^2) + \sum_{l=1}^3 \bar{\psi}_{l+}[i\gamma^\mu\partial_\mu - m_l + gT^a\gamma^\mu Q_{\mu+}^a]\psi_{l+} \right. \\
& \left. - \sum_{l=1}^3 \bar{\psi}_{l-}[i\gamma^\mu\partial_\mu - m_l + gT^a\gamma^\mu Q_{\mu-}^a]\psi_{l-} + \bar{\Psi}_+[i\gamma^\mu\partial_\mu - M + gT^a\gamma^\mu Q_{\mu+}^a]\Psi_+ \right. \\
& \left. - \bar{\Psi}_-[i\gamma^\mu\partial_\mu - M + gT^a\gamma^\mu Q_{\mu-}^a]\Psi_-] \right] \\
& \times \langle Q_+, \psi_+^u, \bar{\psi}_+^u, \psi_+^d, \bar{\psi}_+^d, \psi_+^s, \bar{\psi}_+^s, \Psi_+, \bar{\Psi}_+, 0 | \rho | 0, \Psi_-, \bar{\Psi}_-, \bar{\psi}_-^s, \psi_-^s, \bar{\psi}_-^d, \psi_-^d, \bar{\psi}_-^u, \psi_-^u, Q_- \rangle
\end{aligned} \tag{78}$$

where $G_f^a(Q)$ is given by eq. (66) and eq. (62) [by using eq. (61)] becomes eq. (67) which for finite transformation gives eq. (68). Changing the integration variable from unprimed variable to primed variable we find from eq. (78)

$$\begin{aligned}
& \langle in | \bar{\Psi}_r(x_1)\Psi_r(x_1)\bar{\Psi}_s(x_2)\Psi_s(x_2) | in \rangle_A \\
& = \int [dQ'_+][dQ'_-][d\bar{\psi}'_{1+}][d\bar{\psi}'_{1-}][d\psi'_{1+}][d\psi'_{1-}][d\bar{\psi}'_{2+}][d\bar{\psi}'_{2-}][d\psi'_{2+}][d\psi'_{2-}][d\bar{\psi}'_{3+}][d\bar{\psi}'_{3-}][d\psi'_{3+}][d\psi'_{3-}]
\end{aligned}$$

$$\begin{aligned}
& [d\bar{\Psi}'_+][d\bar{\Psi}'_-][d\Psi'_+][d\Psi'_-] \bar{\Psi}'_r(x_1)\Psi'_r(x_1)\bar{\Psi}'_s(x_2)\Psi'_s(x_2) \\
& \times \det\left(\frac{\delta G_f^a(Q'_+)}{\delta\omega_+^b}\right) \times \det\left(\frac{\delta G_f^a(Q'_-)}{\delta\omega_-^b}\right) \times \exp[i \int d^4x [-\frac{1}{4}(F_{\mu\nu}^{a2}[Q'_+] - F_{\mu\nu}^{a2}[Q'_-]) - \frac{1}{2\alpha}((G_f^a(Q'_+))^2 \\
& -(G_f^a(Q'_-))^2) + \sum_{l=1}^3 \bar{\psi}'_{l+}[i\gamma^\mu\partial_\mu - m_l + gT^a\gamma^\mu Q'_{\mu+}] \psi'_{l+} - \sum_{l=1}^3 \bar{\psi}'_{l-}[i\gamma^\mu\partial_\mu - m_l + gT^a\gamma^\mu Q'_{\mu-}] \psi'_{l-} \\
& + \bar{\Psi}'_+[i\gamma^\mu\partial_\mu - M + gT^a\gamma^\mu Q'_{\mu+}] \Psi'_+ - \bar{\Psi}'_-[i\gamma^\mu\partial_\mu - M + gT^a\gamma^\mu Q'_{\mu-}] \Psi'_-]] \\
& \times \langle Q'_+, \psi'_+, \bar{\psi}'_+, \psi'^d_+, \bar{\psi}'^d_+, \psi'^s_+, \bar{\psi}'^s_+, \Psi'_+, \bar{\Psi}'_+, 0 | \rho | 0, \Psi'_-, \bar{\Psi}'_-, \bar{\psi}'^s_-, \psi'^s_-, \bar{\psi}'^d_-, \psi'^d_-, \bar{\psi}'^u_-, \psi'^u_-, Q'_- \rangle.
\end{aligned} \tag{79}$$

This is because a change of integration variable from unprimed variable to primed variable does not change the value of the integration. Note that since we are working in the frozen ghost formalism at the initial time [53, 54] the $\langle Q_+, \psi_+, \bar{\psi}_+, \psi^d_+, \bar{\psi}^d_+, \psi^s_+, \bar{\psi}^s_+, \Psi_+, \bar{\Psi}_+, 0 | \rho | 0, \Psi_-, \bar{\Psi}_-, \bar{\psi}^s_-, \psi^s_-, \bar{\psi}^d_-, \psi^d_-, \bar{\psi}^u_-, \psi^u_-, Q_- \rangle$ in eq. (3) corresponding to initial density of state in non-equilibrium QCD is gauge invariant by definition. Hence from eqs. (68) and (69) we find

$$\begin{aligned}
& \langle Q'_+, \psi'_+, \bar{\psi}'_+, \psi'^d_+, \bar{\psi}'^d_+, \psi'^s_+, \bar{\psi}'^s_+, \Psi'_+, \bar{\Psi}'_+, 0 | \rho | 0, \Psi'_-, \bar{\Psi}'_-, \bar{\psi}'^s_-, \psi'^s_-, \bar{\psi}'^d_-, \psi'^d_-, \bar{\psi}'^u_-, \psi'^u_-, Q'_- \rangle \\
& = \langle Q_+, \psi_+, \bar{\psi}_+, \psi^d_+, \bar{\psi}^d_+, \psi^s_+, \bar{\psi}^s_+, \Psi_+, \bar{\Psi}_+, 0 | \rho | 0, \Psi_-, \bar{\Psi}_-, \bar{\psi}^s_-, \psi^s_-, \bar{\psi}^d_-, \psi^d_-, \bar{\psi}^u_-, \psi^u_-, Q_- \rangle.
\end{aligned} \tag{80}$$

We follow the similar procedure that was employed to prove factorization theorem of j/ψ production in QCD in vacuum in the previous section. When background field $A^{\mu a}(x)$ is the SU(3) pure gauge as given by eq. (55) then by using eqs. (71) and (80) in eq. (79) we find

$$\begin{aligned}
& \langle in | \bar{\Psi}_r(x_1)\Psi_r(x_1)\bar{\Psi}_s(x_2)\Psi_s(x_2) | in \rangle_A = \\
& = \int [dQ_+][dQ_-][d\bar{\psi}_{1+}][d\bar{\psi}_{1-}][d\psi_{1+}][d\psi_{1-}][d\bar{\psi}_{2+}][d\bar{\psi}_{2-}][d\psi_{2+}][d\psi_{2-}][d\bar{\psi}_{3+}][d\bar{\psi}_{3-}][d\psi_{3+}][d\psi_{3-}] \\
& [d\bar{\Psi}_+][d\bar{\Psi}_-][d\Psi_+][d\Psi_-] \bar{\Psi}_r(x_1)\Psi_r(x_1)\bar{\Psi}_s(x_2)\Psi_s(x_2) \times \det\left(\frac{\delta\partial_\mu Q_+^{\mu a}}{\delta\omega_+^b}\right) \times \det\left(\frac{\delta\partial_\mu Q_-^{\mu a}}{\delta\omega_-^b}\right) \\
& \exp[i \int d^4x [-\frac{1}{4}(F_{\mu\nu}^{a2}[Q_+] - F_{\mu\nu}^{a2}[Q_-]) - \frac{1}{2\alpha}((\partial_\mu Q_+^{\mu a})^2 - (\partial_\mu Q_-^{\mu a})^2) \\
& + \sum_{l=1}^3 \bar{\psi}_{l+}[i\gamma^\mu\partial_\mu - m_l + gT^a\gamma^\mu Q_{\mu+}] \psi_{l+} - \sum_{l=1}^3 \bar{\psi}_{l-}[i\gamma^\mu\partial_\mu - m_l + gT^a\gamma^\mu Q_{\mu-}] \psi_{l-} \\
& + \bar{\Psi}_+[i\gamma^\mu\partial_\mu - M + gT^a\gamma^\mu Q_{\mu+}] \Psi_+ - \bar{\Psi}_-[i\gamma^\mu\partial_\mu - M + gT^a\gamma^\mu Q_{\mu-}] \Psi_-]] \\
& \times \langle Q_+, \psi_+, \bar{\psi}_+, \psi^d_+, \bar{\psi}^d_+, \psi^s_+, \bar{\psi}^s_+, \Psi_+, \bar{\Psi}_+, 0 | \rho | 0, \Psi_-, \bar{\Psi}_-, \bar{\psi}^s_-, \psi^s_-, \bar{\psi}^d_-, \psi^d_-, \bar{\psi}^u_-, \psi^u_-, Q_- \rangle.
\end{aligned} \tag{81}$$

From eqs. (75) and (81) we find

$$\langle in | \bar{\Psi}_r(x_1) \Psi_r(x_1) \bar{\Psi}_s(x_2) \Psi_s(x_2) | in \rangle = \langle in | \bar{\Psi}_r(x_1) \Psi_r(x_1) \bar{\Psi}_s(x_2) \Psi_s(x_2) | in \rangle_A \quad (82)$$

which proves factorization of infrared divergences at all order in coupling constant in non-equilibrium QCD. Eq. (82) is valid in covariant gauge, in light-cone gauge, in general axial gauges, in general non-covariant gauges and in general Coulomb gauge etc. respectively [28].

From eq. (82) we find that the nonequilibrium-nonperturbative matrix element for j/ψ production from $c\bar{c}$ pair in color singlet and spin triplet state in non-equilibrium QCD in eq. (15) as given by

$$\langle in | \mathcal{O}_H | in \rangle = \langle in | \chi^\dagger \sigma^i \psi a_H^\dagger a_H \psi^\dagger \sigma^i \chi | in \rangle \quad (83)$$

is consistent with factorization theorem of j/ψ production in non-equilibrium QCD at RHIC and LHC at all order in coupling constant.

Eq. (82) proves the factorization theorem of j/ψ production in non-equilibrium QCD at RHIC and LHC at all order in coupling constant where $\langle in | \bar{\Psi}_r(x_1) \Psi_r(x_1) \bar{\Psi}_s(x_2) \Psi_s(x_2) | in \rangle$ is the heavy quark-antiquark gauge invariant nonequilibrium-nonperturbative correlation function in QCD and $\langle in | \bar{\Psi}_r(x_1) \Psi_r(x_1) \bar{\Psi}_s(x_2) \Psi_s(x_2) | in \rangle_A$ is the corresponding heavy quark-antiquark gauge invariant nonequilibrium-nonperturbative correlation function in QCD in the presence of light-like Wilson line. From eq. (82) we find that the infrared divergences due to the soft gluons exchange between charm quark and the nearby light-like quark (or gluon) in non-equilibrium QCD at RHIC and LHC exactly cancel with the corresponding infrared divergences due to the soft gluons exchange between anticharm quark and the same nearby light-like quark (or gluon) at all order in coupling constant in the j/ψ production from color singlet $c\bar{c}$ pair in non-equilibrium QCD. This proves the factorization theorem of j/ψ production in non-equilibrium QCD at RHIC and LHC at all order in coupling constant.

VIII. CONCLUSIONS

J/ψ suppression/production is one of the main signature of quark-gluon plasma detection at RHIC and LHC. In order to study j/ψ suppression/production in high energy heavy-ion collisions at RHIC and LHC, one needs to prove the factorization theorem of j/ψ production

in non-equilibrium QCD medium, otherwise one will predict infinite cross section of j/ψ . In this paper we have proved factorization theorem of j/ψ production in non-equilibrium QCD at RHIC and LHC at all order in coupling constant. This proof is necessary to study the detection of quark-gluon plasma [1, 2, 55, 56] at RHIC and LHC by using j/ψ suppression/production as its signature [8].

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